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Non-linear flexural waves in thin layers

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Abstract

Non-linear flexural waves in thin plates or layers have been analyzed in this paper. The equation of motion of the plate is derived assuming that the motion is antisymmetric about the mid-plane of the plate and that the plate is thin. The plate is considered to be elastic. The Von Karman non-linear strains and Landau elastic constants have been used to model geometric and material non-linearities, respectively. An asymptotic analysis of wave motion is presented using the method of multiple scales. Evolution equations are derived for small amplitude traveling flexural elastic waves. Numerical results show waveform distortion, amplitude amplification, and harmonic generation.

1. Introduction

Recently, non-destructive evaluation and material characterization using non-linear waves have received attention because of potential industrial applications. Material and geometric non-linearities are often necessary to consider for modelling structures when the excitation amplitude is large. These non-linearities affect the elastic wave dispersion and attenuation. Experimentally, material and geometric non-linearities have been observed in waveform distortion, wave-amplitude amplification and harmonic generation [1,2]. In order to model these effects, a non-linear quantitative analysis must be developed. In linear wave theory, guided modes in a plate at a fixed frequency are linearly independent, whereas material and geometric non-linearities induce interaction of the modes. This is exhibited by harmonic generation and the subject of this investigation. Early work on non-linear wave interaction was by Göldberg [3], who studied bulk wave interaction in elastic solids and harmonic generation. An asymptotic solution using a regular

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expansion has been used by Jones and Kobett [4]. This captured only quadratic harmonic generation due to quadratic non-linearity. Asymptotic solutions to non-linear equations have also been obtained using perturbation methods [5,6]. The method of multiple scales is an efficient tool for obtaining solutions to non-linear equations. This technique is based upon a perturbation of the linear solution. The solvability conditions and the necessary evolution equations for the amplitude dependence on the multiple space and time scales [7–10] are easily obtained. This method has been employed previously to study non-linear surface waves by Kalyanasundaram [11,12], Lardner [13–16], and Harvey and Tupholme [17,18]. In their work, by solving the evolution equations, they conclude that the fundamental harmonic amplitude grows initially and then remains bounded. It is found that wave coupling causes waveform distortion and harmonic generation. Non-linear surface waves are also studied using the averaging of the Hamiltonian by Zabolotskaya [19] and by Hamilton et al. [20] for isotropic and cubic materials. In their work, the evolution equations governing wave amplitude were derived using Fourier series expansion. Also, Fu and Hill [21] employed the method of multiple scales to study weakly dispersive surface waves in a coated half-space. It appears that non-linear guided waves in thin plates have not received much attention. A relevant work on non-linear guided waves in elastic isotropic plates is by Fu [22]. He found a dramatic influence of material and geometric non-linearities on the amplitude of weakly dispersive traveling waves in plates. Non-linear waves in elastic bars were studied by Kovriguine and Potapov [23], and Kovriguine [24]. They used the classical beam theory to study non-linear interactions between longitudinal and flexural waves in rectangular bars.

In this work, non-linear flexural waves in infinite plates are modelled using the first order Mindlin plate theory. Material and geometric non-linearities are included using the third order elastic constants and Von Karman non-linear strains [25], respectively. The method of multiple scales is used to derive the amplitude evolution equations of self-modulated non-linear flexural waves.

2. Mathematical formulation

2.1. Equations of motion

For thin plates, the displacement components for antisymmetric motion about the mid-plane can be approximated as

$$\begin{aligned}\tilde{u}(x,y,z,t) &= z\psi(x,y,t),\\ \tilde{v}(x,y,z,t) &= z\phi(x,y,t),\\ \tilde{w}(x,y,z,t) &= w(x,y,t),\end{aligned} \tag{1}$$

where w is the displacement component along the z-axis, which is normal to the mid-plane, and (ϕ, ψ) are the rotations of the normal to the mid-plane about x and y axes, respectively. For large deformation, the Lagrangian representation of the strain–displacement relation is

$$E_{ij} = \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} + \frac{\partial \tilde{u}_k}{\partial x_i}\frac{\partial \tilde{u}_k}{\partial x_j}\right). \tag{2}$$

Adopting Von Karman non-linear strain components [26], i.e., quadratic in w and linear in ϕ and ψ yields,

$$E_{xx} = z\psi_{,x} + w_{,x}^2/2, \quad E_{yz} = k_1(\phi + w_{,y})/2,$$
$$E_{yy} = z\phi_{,y} + w_{,y}^2/2, \quad E_{zx} = k_2(\psi + w_x)/2,$$
$$E_{zz} = -\beta(E_{xx} + E_{yy}), \quad E_{xy} = (z(\psi_{,y} + \phi_{,x}) + w_{,x}w_{,y}))/2, \tag{3}$$

where $k_1 = k_2 = \pi/\sqrt{12}$, are the shear correction factors, and $\beta = \lambda/(2\mu + \lambda)$ is introduced to account for plane stress condition. Assuming that the material is isotropic and elastic, the constitutive equation for the Cauchy stress tensor is [27]

$$\sigma_{ij} = \lambda E_{kk}\delta_{ij} + 2\mu E_{ij} + \delta_{ij}(CE_{kk}E_{ll} + BE_{kl}E_{lk}) + 2BE_{kk}E_{ij} + AE_{jk}E_{ki}, \tag{4}$$

where A, B, and C are the third order elastic constants. The corresponding first Piola–Kirchhoff stress tensor can be expressed as

$$\Sigma_{ij} = \left(\delta_{ik} + \frac{\partial u_i}{\partial x_k}\right)\sigma_{kj}. \tag{5}$$

In the absence of any body forces, the equations of motion are

$$\frac{\partial \Sigma_{xx}}{\partial x} + \frac{\partial \Sigma_{xy}}{\partial y} + \frac{\partial \Sigma_{xz}}{\partial z} = \rho \frac{\partial^2 \tilde{u}}{\partial t^2},$$
$$\frac{\partial \Sigma_{yx}}{\partial x} + \frac{\partial \Sigma_{yy}}{\partial y} + \frac{\partial \Sigma_{yz}}{\partial z} = \rho \frac{\partial^2 \tilde{v}}{\partial t^2},$$
$$\frac{\partial \Sigma_{zx}}{\partial x} + \frac{\partial \Sigma_{zy}}{\partial y} + \frac{\partial \Sigma_{zz}}{\partial z} = \rho \frac{\partial^2 \tilde{w}}{\partial t^2}. \tag{6}$$

In order to model the antisymmetric motion about the mid-plane of the plate, the third equation and z-times the first two equations are integrated with respect to z from $-h$ to $+h$ giving the following plate equations of motion:

$$M_{xx,x} + M_{xy,y} - Q_{xz} = \frac{2\rho h^3}{3}\ddot{\psi},$$
$$M_{yx,x} + M_{yy,y} - Q_{yz} = \frac{2\rho h^3}{3}\ddot{\phi},$$
$$Q_{zx,x} + Q_{zy,y} = 2\rho h\ddot{w}, \tag{7}$$

where

$$(Q_{xz}, Q_{yz}, Q_{zx}, Q_{zy}) = \int_{-h}^{+h} (\Sigma_{xz}, \Sigma_{yz}, \Sigma_{zx}, \Sigma_{zy})\,\mathrm{d}z,$$

$$(M_{xx}, M_{yy}, M_{xy}, M_{yx}) = \int_{-h}^{+h} z(\Sigma_{xx}, \Sigma_{yy}, \Sigma_{xy}, \Sigma_{yx})\,\mathrm{d}z.$$

Assuming a plane wave propagating along the x-axis of the thin plate, i.e., $\partial/\partial y = 0$, the equations of motion may be simplified as

$$\begin{aligned}
&\frac{2h^3}{3}(2\mu+\bar{\lambda})\psi_{,xx} - 2hk_2^2\mu(\psi + w_{,x}) + f_1^{NL} = \frac{2\rho h^3}{3}\ddot{\psi},\\
&\frac{2h^3}{3}\mu\phi_{,xx} - 2hk_1^2\mu\phi + f_2^{NL} = \frac{2\rho h^3}{3}\ddot{\phi},\\
&2hk_2^2\mu(\psi_{,x} + w_{,xx}) + f_3^{NL} = 2\rho h\ddot{w},
\end{aligned} \tag{8}$$

where the subscript $()_{,x}$ represents $\partial/\partial x$, and the dot stands for $\partial/\partial t$. The non-linear terms f_i^{NL} are

$$\begin{aligned}
f_1^{NL} &= a_1(w_{,x}^2\psi_{,x})_{,x} - a_2\psi w_{,x}^2 - a_3 w_{,x}^3,\\
f_2^{NL} &= b_1(w_{,x}^2\phi_{,x})_{,x} - b_2\phi w_{,x}^2,\\
f_3^{NL} &= (c_1 w_{,x}^3 + c_2\psi w_{,x}^2)_{,x}.
\end{aligned} \tag{9}$$

Here, the coefficients a_i, b_i, and c_i are functions of material properties and plate thickness:

$$a_1 = \frac{h^3}{3}(\bar{\lambda} + 2\mu + 5A/2 + B(7 - 4\beta + 2\beta^2) + 2C(1-\beta)^2),$$

$$a_2 = h(A/2 + B)(2-\beta), \quad a_3 = h(1-\beta)(A/2+B),$$

$$b_1 = \frac{h^3}{3}(\bar{\lambda} + 2\mu + A + B(2-\beta)), \quad b_2 = h\left(\frac{A}{2}(1-\beta) + B(2-\beta)\right),$$

$$c_1 = h(\bar{\lambda} + 2\mu + (A/2+B)(2-\beta)), \quad c_2 = h(3-\beta)(A/2+B),$$

$$\beta = \frac{\lambda}{2\mu+\lambda}.$$

Note that $\bar{\lambda} = \lambda(1-\beta)$.

2.2. Normalization

In order to carry out a perturbation analysis, these equations are cast in a non-dimensional form. For this purpose, the following parameters are introduced;

$$\hat{x} = \frac{x}{h}, \quad \hat{t} = \frac{t}{t_s}, \quad \hat{\psi} = \psi, \quad \hat{\phi} = \phi, \quad \hat{w} = \frac{w}{h},$$

$$C_s^2 = \frac{\mu}{\rho}, \quad C_d^2 = \frac{\bar{\lambda} + 2\mu}{\rho}, \quad t_s = \frac{h}{C_s}, \quad K = kh, \quad \Omega = \frac{\omega h}{C_s}.$$

Here, $\hat{x}$ and $\hat{t}$ are the non-dimensional space and time variables. $\hat{\psi}, \hat{\phi}$, and $\hat{w}$ are the non-dimensional components of the rotations around x- and y-axis, and the displacement along z-axis, respectively. C_d and C_s are, respectively, the dilatational and shear wave speeds, t_s the shear wave travel time through the thickness, K and Ω the non-dimensional wavenumber and circular wave frequency, respectively. The resulting non-dimensional form of Eq. (8) is

$$\begin{aligned}
&\frac{C_d^2}{C_s^2}\hat{\psi}'' - 3k_2^2(\hat{\psi} + \hat{w}') + \hat{f}_1^{NL} = \ddot{\hat{\psi}},\\
&\hat{\phi}'' - 3k_1^2\hat{\phi} + \hat{f}_2^{NL} = \ddot{\hat{\phi}},\\
&k_2^2(\hat{\psi}' + \hat{w}'') + \hat{f}_3^{NL} = \ddot{\hat{w}},
\end{aligned} \tag{10}$$

where the prime represents $\partial/\partial\hat{x}$ and the dot $\partial/\partial\hat{t}$. The non-linear terms $\hat{f}_i^{NL}$ are

$$\begin{aligned}
&\hat{f}_1^{NL} = \hat{a}_1(\hat{w}'^2\hat{\psi}')' - \hat{a}_2\hat{\psi}\hat{w}'^2 - \hat{a}_3\hat{w}'^3,\\
&\hat{f}_2^{NL} = \hat{b}_1(\hat{w}'^2\hat{\phi}')' - \hat{b}_2\hat{\phi}\hat{w}'^2,\\
&\hat{f}_3^{NL} = (\hat{c}_1\hat{w}'^3 + \hat{c}_2\hat{\psi}\hat{w}'^2)',
\end{aligned} \tag{11}$$

which include non-dimensional parameters as follows:

$$\hat{a}_1 = \frac{1}{2\mu}(\bar{\lambda} + 2\mu + 5A/2 + B(7 - 4\beta + 2\beta^2) + 2C(1-\beta)^2),$$

$$\hat{a}_2 = \frac{3}{2\mu}(A/2 + B)(2 - \beta), \quad \hat{a}_3 = \frac{3}{2\mu}(1-\beta)(A/2 + B),$$

$$\hat{b}_1 = \frac{1}{2\mu}(\bar{\lambda} + 2\mu + A + B(2-\beta)), \quad \hat{b}_2 = \frac{3}{2\mu}\left(\frac{A}{2}(1-\beta) + B(2-\beta)\right),$$

$$\hat{c}_1 = \frac{1}{2\mu}(\bar{\lambda} + 2\mu + (A/2 + B)(2-\beta)), \quad \hat{c}_2 = \frac{1}{2\mu}(3-\beta)(A/2 + B).$$

Eqs. (10) represent the non-dimensional form of the antisymmetric wave motion. These equations are coupled and the cubic non-linearity makes it difficult to solve them analytically. Here, the method of multiple scales is used to solve the equations.

2.3. The method of multiple scales

A perturbation solution is sought around the linear solution using the method of multiple scales. This leads to equations that depict the non-linearity influence on the wave amplitude, wavenumber and frequency. The independent variables are expanded as [5,6]

$$T_n = \varepsilon^n\hat{t}, \quad X_n = \varepsilon^n\hat{x}, \tag{12}$$

where T_n and X_n are the slow time and space scales, ε is a small expansion parameter (<1) and measures the order of the non-linearity. In order to capture the cubic non-linearity of the flexural wave equations given by Eq. (10), one represents the derivatives with respect to multiple space and time variables as

$$\begin{aligned}
&\frac{\partial}{\partial\hat{t}} = \frac{\partial}{\partial T_0} + \varepsilon^2\frac{\partial}{\partial T_2} + \dots,\\
&\frac{\partial}{\partial\hat{x}} = \frac{\partial}{\partial X_0} + \varepsilon^2\frac{\partial}{\partial X_2} + \dots\ .
\end{aligned} \tag{13}$$

Eqs. (13) are the derivative expansions. Similarly, expanding the dependent variables in the following form will result in solutions of different orders:

$$\hat{\psi} = \varepsilon\psi_1 + \varepsilon^3\psi_3 + \dots,$$
$$\hat{\phi} = \varepsilon\phi_1 + \varepsilon^3\phi_3 + \dots,$$
$$\hat{w} = \varepsilon w_1 + \varepsilon^3 w_3 + \dots \quad . \tag{14}$$

Here, the first terms in the expansion correspond to the linear solution, and the subsequent terms are the corrections. Substituting Eqs. (13) and (14) in the non-linear system (10), yields the following systems:

$$O(\varepsilon):\quad \frac{C_d^2}{C_s^2}\frac{\partial^2\psi_1}{\partial X_0^2} - 3k_2^2\left(\psi_1 + \frac{\partial w_1}{\partial X_0}\right) - \frac{\partial^2\psi_1}{\partial T_0^2} = 0,$$
$$\frac{\partial^2\phi_1}{\partial X_0^2} - 3k_1^2\phi_1 - \frac{\partial^2\phi_1}{\partial T_0^2} = 0,$$
$$k_2^2\left(\frac{\partial\psi_1}{\partial X_0} + \frac{\partial^2 w_1}{\partial X_0^2}\right) - \frac{\partial^2 w_1}{\partial T_0^2} = 0, \tag{15}$$

and

$$O(\varepsilon^3):\quad \frac{C_d^2}{C_s^2}\frac{\partial^2\psi_3}{\partial X_0^2} - 3k_2^2\left(\psi_3 + \frac{\partial w_3}{\partial X_0}\right) - \frac{\partial^2\psi_3}{\partial T_0^2}$$
$$= -2\frac{C_d^2}{C_s^2}\frac{\partial^2\psi_1}{\partial X_0\partial X_2} + 2k_2^2\frac{\partial w_1}{\partial X_2} - \hat{a}_1\frac{\partial}{\partial X_0}\left(\frac{\partial\psi_1}{\partial X_0}\left(\frac{\partial w_1}{\partial X_0}\right)^2\right)$$
$$+\ \hat{a}_2\psi_1\left(\frac{\partial w_1}{\partial X_0}\right)^2 + \hat{a}_3\left(\frac{\partial w_1}{\partial X_0}\right)^3 + 2\frac{\partial^2\psi_1}{\partial T_0\partial T_2},$$
$$\frac{\partial^2\phi_3}{\partial X_0^2} - 3k_1^2\phi_3 - \frac{\partial^2\phi_3}{\partial T_0^2} = 2\frac{\partial^2\phi_1}{\partial X_0\partial X_2} - \hat{b}_1\frac{\partial}{\partial X_0}\left(\frac{\partial\phi_1}{\partial X_0}\left(\frac{\partial w_1}{\partial X_0}\right)^2\right)$$
$$+\ \hat{b}_2\phi_1\left(\frac{\partial w_1}{\partial X_0}\right)^2 + 2\frac{\partial^2\phi_1}{\partial T_0\partial T_2},$$
$$k_2^2\left(\frac{\partial\psi_3}{\partial X_0} + \frac{\partial^2 w_3}{\partial X_0^2}\right) - \frac{\partial^2 w_3}{\partial T_0^2} = -k_2^2\left(\frac{\partial\psi_1}{\partial X_2} + \frac{\partial^2 w_1}{\partial X_0\partial X_2}\right)$$
$$-\frac{\partial}{\partial X_0}\left(\hat{c}_1\left(\frac{\partial w_1}{\partial X_0}\right)^3 + \hat{c}_2\psi_1\left(\frac{\partial w_1}{\partial X_0}\right)^2\right) + 2\frac{\partial^2 w_1}{\partial T_0\partial T_2}. \tag{16}$$

The first order system ($O(\varepsilon)$) represents the linear equation of antisymmetric wave motion in the plate. The third order system ($O(\varepsilon^3)$) represents the next order correction to the linear system. Note that the right-hand side of Eq. (16) is known from Eq. (15) and will act as a forcing term representing the non-linear interaction. The homogeneous solution to this higher order system will have the same form as the linear solution. Here, one is concerned with the particular solution of the higher order system. The monoharmonic solution of Eq. (15) for the first flexural mode may be written as

$$\begin{pmatrix} \psi_1 \\ \phi_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} v_1(X_2, T_2) \\ v_2(X_2, T_2) \\ v_3(X_2, T_2) \end{pmatrix} e^{i(K_0 X_0 - \Omega_0 T_0)} + c.c. \tag{17}$$

Clearly, the coefficients are functions of the slower variables X_2 and T_2. Substituting Eq. (17) in the right-hand side of Eq. (16) yields a non-homogeneous system having a particular solution, which is the correction solution due to non-linearity. Therefore, the presence of the monoharmonic force in the right-hand side will give secular terms, which will grow without bound. These terms are the coefficients of $e^{i(K_0 X_0 - \Omega_0 T_0)}$ on the right-hand side of Eq. (16). In order to satisfy the solvability condition, it is necessary to equate the secular terms to zero. Thus, Eq. (16) may be reduced to

$$\begin{aligned}
&\frac{C_d^2}{C_s^2}\frac{\partial^2 \psi_3}{\partial X_0^2} - 3k_2^2\left(\psi_3 + \frac{\partial w_3}{\partial X_0}\right) - \frac{\partial^2 \psi_3}{\partial T_0^2} \\
&\quad = -(3\hat{a}_1 K_0^4 v_3^2 v_1 + \hat{a}_2 K_0^2 v_3^2 v_1 + i\hat{a}_3 K_0^3 v_3^3) e^{3i(K_0 X_0 - \Omega_0 T_0)}, \\
&\frac{\partial^2 \phi_3}{\partial X_0^2} - 3k_1^2 \phi_3 - \frac{\partial^2 \phi_3}{\partial T_0^2} = -(3\hat{b}_1 K_0^4 v_3^2 v_2 + \hat{b}_2 K_0^2 v_3^2 v_2) e^{3i(K_0 X_0 - \Omega_0 T_0)}, \\
&k_2^2\left(\frac{\partial \psi_3}{\partial X_0} + \frac{\partial^2 w_3}{\partial X_0^2}\right) - \frac{\partial^2 w_3}{\partial T_0^2} = -(3\hat{c}_1 K_0^4 v_3^3 - 3i\hat{c}_2 K_0^3 v_3^2 v_1) e^{3i(K_0 X_0 - \Omega_0 T_0)},
\end{aligned} \tag{18}$$

and the equations obtained by setting the secular terms to zero,

$$\begin{aligned}
&-2iK_0 \frac{C_d^2}{C_s^2}\frac{\partial v_1}{\partial X_2} + 3k_2^2 \frac{\partial v_3}{\partial X_2} + \hat{a}_1 K_0^4 (2|v_3|^2 v_1 + v_3^2 \bar{v}_1) \\
&\quad + \hat{a}_2 K_0^2 (2|v_3|^2 v_1 - v_3^2 \bar{v}_1) + 3iK_0^3 \hat{a}_3 |v_3|^2 v_3 - 2i\Omega_0 \frac{\partial v_1}{\partial T_2} = 0, \\
&-2iK_0 \frac{\partial v_2}{\partial X_2} + \hat{b}_1 K_0^4 (2|v_3|^2 v_2 + v_3^2 \bar{v}_2) \\
&\quad + \hat{b}_2 K_0^2 (2|v_3|^2 v_2 - v_3^2 \bar{v}_2) - 2i\Omega_0 \frac{\partial v_2}{\partial T_2} = 0, \\
&-k_2^2\left(\frac{\partial v_1}{\partial X_2} + 2iK_0 \frac{\partial v_3}{\partial X_2}\right) + 3\hat{c}_1 K_0^4 |v_3|^2 v_3 \\
&\quad - i\hat{c}_2 K_0^3 (2|v_3|^2 v_1 - v_3^2 \bar{v}_1) - 2i\Omega_0 \frac{\partial v_3}{\partial T_2} = 0.
\end{aligned} \tag{19}$$

Non-homogeneous system (18) is the correction system describing the harmonic generation. Furthermore, the secular equations (19) are the amplitude evolution equations showing the wave amplitude dependence on the slow variables X_2 and T_2. The general solution of Eq. (18) may be written in the form:

$$\begin{pmatrix} \psi_3 \\ \phi_3 \\ w_3 \end{pmatrix} = \begin{pmatrix} q_1(X_2, T_2) \\ q_2(X_2, T_2) \\ q_3(X_2, T_2) \end{pmatrix} e^{3i(K_0 X_0 - \Omega_0 T_0)} + c.c, \tag{20}$$

where $(q_1 \;\; q_2 \;\; q_3)^T$ can be obtained by solving the following algebraic equation:

$$\begin{bmatrix} \left(9\dfrac{C_d^2}{C_s^2}K_0^2 + 3k_2^2 - 9\Omega_0^2\right) & 0 & 9ik_2^2K_0 \\ 0 & (9K_0^2 + 3k_2^2 - 9\Omega_0^2) & 0 \\ -3ik_2^2K_0 & 0 & (9k_2^2K_0^2 - 9\Omega_0^2) \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = K_0^2 v_3^2 \begin{pmatrix} (2\hat{a}_1 K_0^2 v_1 + \hat{a}_2 v_1 + i\hat{a}_3 K_0 v_3) \\ (3\hat{b}_1 K_0^2 v_2 + \hat{b}_2 v_2) \\ (\hat{c}_1 K_0^2 v_3 - 3i\hat{c}_2 K_0 v_1) \end{pmatrix}. \tag{21}$$

Solving for q_n in terms of v_n will determine the total solution, which can be written in the following non-dimensional form as

$$\begin{pmatrix} \hat{\psi}(\hat{x}, \hat{t}) \\ \hat{\phi}(\hat{x}, \hat{t}) \\ \hat{w}(\hat{x}, \hat{t}) \end{pmatrix} = \varepsilon \begin{pmatrix} v_1(\varepsilon^2\hat{x}, \varepsilon^2\hat{x}) \\ v_2(\varepsilon^2\hat{x}, \varepsilon^2\hat{x}) \\ v_3(\varepsilon^2\hat{x}, \varepsilon^2\hat{x}) \end{pmatrix} e^{i(K_0\hat{x} - \Omega_0\hat{t})} + \varepsilon^3 \begin{pmatrix} q_1(\varepsilon^2\hat{x}, \varepsilon^2\hat{t}) \\ q_2(\varepsilon^2\hat{x}, \varepsilon^2\hat{t}) \\ q_3(\varepsilon^2\hat{x}, \varepsilon^2\hat{t}) \end{pmatrix} e^{3i(K_0\hat{x} - \Omega_0\hat{t})}. \tag{22}$$

Eq. (22) shows the non-linearity influence on the flexural wave motion. The evolution equations have to be solved numerically to study the non-linearity effects.

3. Numerical results and discussion

Eqs. (19)–(21) depend on the third order elastic constants as well as on the wavenumber. The bi-quadratic dependence on the wavenumber implies that non-linearity will be more significant for high-frequency wave propagation, i.e., $K_0 > 1$. For low-frequency measurements of flexural waves, the non-linearity can be negligible and the correction solution asymptotically decays and the wave amplitudes v_n remain constant. Here, Eqs. (19) are solved numerically using the method of lines, i.e., solving for space at each time step. Explicit finite difference scheme has been used to solve the evolution equations. In this work a thin plate of aluminum is considered with Lamé constants $\mu = 27$ GPa and $\lambda = 57$ GPa, and mass density $\rho = 2727\ \mathrm{kg/m^3}$. Landau third order

Table 1
Non-linear coefficients

Non-linearity	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_3$	$\hat{b}_1$	$\hat{b}_2$	$\hat{c}_1$	$\hat{c}_2$
Geometric	1.51	0	0	1.51	0	1.51	0
Material	−36.75	−29.73	−9.73	−11.43	−20.84	−9.91	−16.58
Both	−35.24	−29.73	−9.73	−9.92	−20.84	−8.40	−16.58

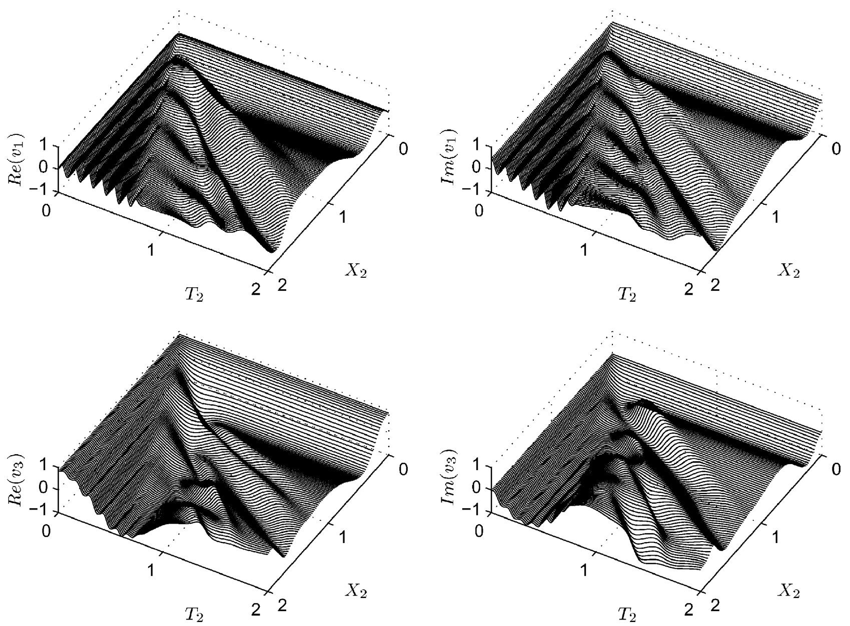

Fig. 1. Real and imaginary parts of the monoharmonic wave amplitude components calculated using Eq. (19) for non-dimensional wavenumber $K_0 = 1.0$ and non-linearity parameter $\varepsilon = 0.05$.

elastic constants are $A = -320$ GPa, $B = -200$ GPa, and $C = -190$ GPa. It is clearly noticeable that the Landau third order elastic constants are generally of larger order of magnitude than the Lamé constants. For aluminum material, the ratio of the Landau third order elastic constants to the Lamé constant μ is of order 10. This suggests that the non-linearity in the constitutive equation (effects of higher-order elastic constants) will dominate over the geometric non-linearity. Table 1 lists the values of the non-linear coefficients due to geometric and/or material non-linearity. Considering geometric non-linearity alone, the non-linear coefficients are positive and small in magnitude compared to those arising from material non-linearity.

Numerical results are presented in Fig. 1 considering both material and geometric non-linearities for monoharmonic wave amplitude v_n. They show the dependence of the wave amplitude on the slow variables (X_2, T_2) at a non-dimensional wavenumber of $K_0 = 1$ and non-linear parameter $\varepsilon = 0.05$. Plots of real and imaginary parts of the tri-harmonic correction

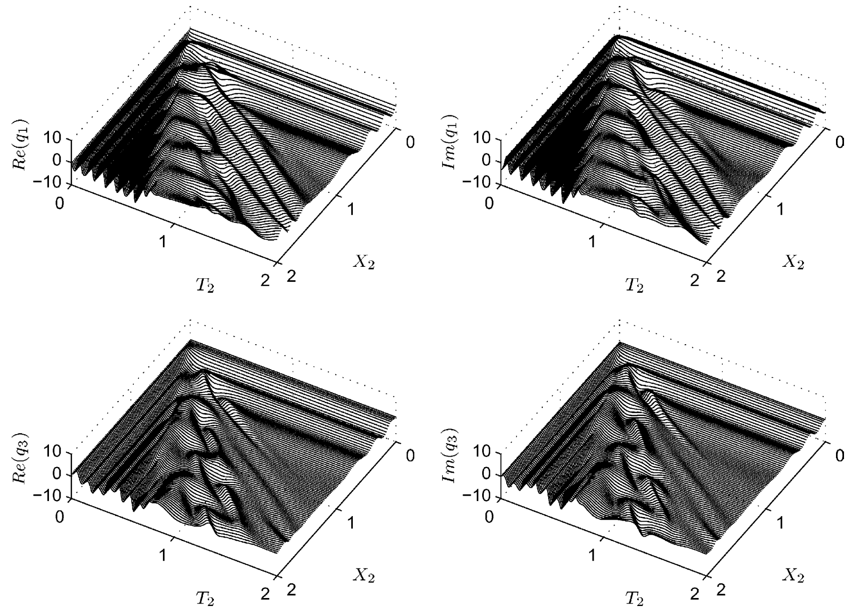

Fig. 2. Real and imaginary parts of the tri-harmonic wave amplitude components calculated using Eq. (21) for non-dimensional wavenumber $K_0 = 1.0$ and non-linearity parameter $\varepsilon = 0.05$.

amplitude q_n are presented in Fig. 2. For flexural wave in isotropic material, transverse components $v_2 = q_2 = 0$. Even for weak non-linearity, i.e., $\varepsilon \ll 1$, the non-linear effects can be clearly seen in the variation of the coefficients of both the monoharmonic and tri-harmonic terms. It is observed that the space and time dependence is quite different for $x - 3t < 0$ than when $x - 3t > 0$. The characteristic line $\approx (x - 3t)$ demarcates the slow-time oscillations to soliton-like behavior of the wave amplitude. This phenomenon is caused by the accumulation of the effects of material and geometric non-linearities on the amplitude over distance and time. Notice that the ratio of the correction solution to linear solution is of $O(\varepsilon^2)$. Both slowly varying wave amplitudes, i.e., (v_i, q_i), determine the wave profile distortion and harmonic generation of the primary wave.

To obtain further understanding of the non-linear flexural wave behavior in time, both real and imaginary parts of the normal displacement and rotation components $(\hat{w}, \hat{\psi})$ are plotted in Figs. 3 and 4 for different observation points ($\hat{x} = 0, 160, 320, 480, 640,$ and 800). At $\hat{x} = 0$ a flexural wave train is seen to propagate in the thin plate with linear wave behavior. For $\hat{x} > 0$ the transverse displacement component $\hat{w}$ shows linear behavior for small time but its amplitude is seen to decrease as t increases. Very small amplitude modulation (beating) is seen at small time for $\hat{x} > 0$. Fig. 4 shows that the rotational component $\hat{\psi}$ has strikingly different behavior when $\hat{x} > 0$. $\hat{\psi}$ component shows pronounced beating for $\hat{t} < \hat{x}/3$. Another feature is that solitons start to appear at $\hat{t} > \hat{x}/3$ and wave amplification is clearly observed in the real and imaginary parts of $\hat{\psi}$ at large times.

Fig. 5 shows the real parts of wave components ($\hat{w}$ and $\hat{\psi}$) calculated at $\hat{x} = 200$ for different values of non-linear parameter ($\varepsilon = 0.01, 0.05, 0.07$). It is clearly seen that for very weak

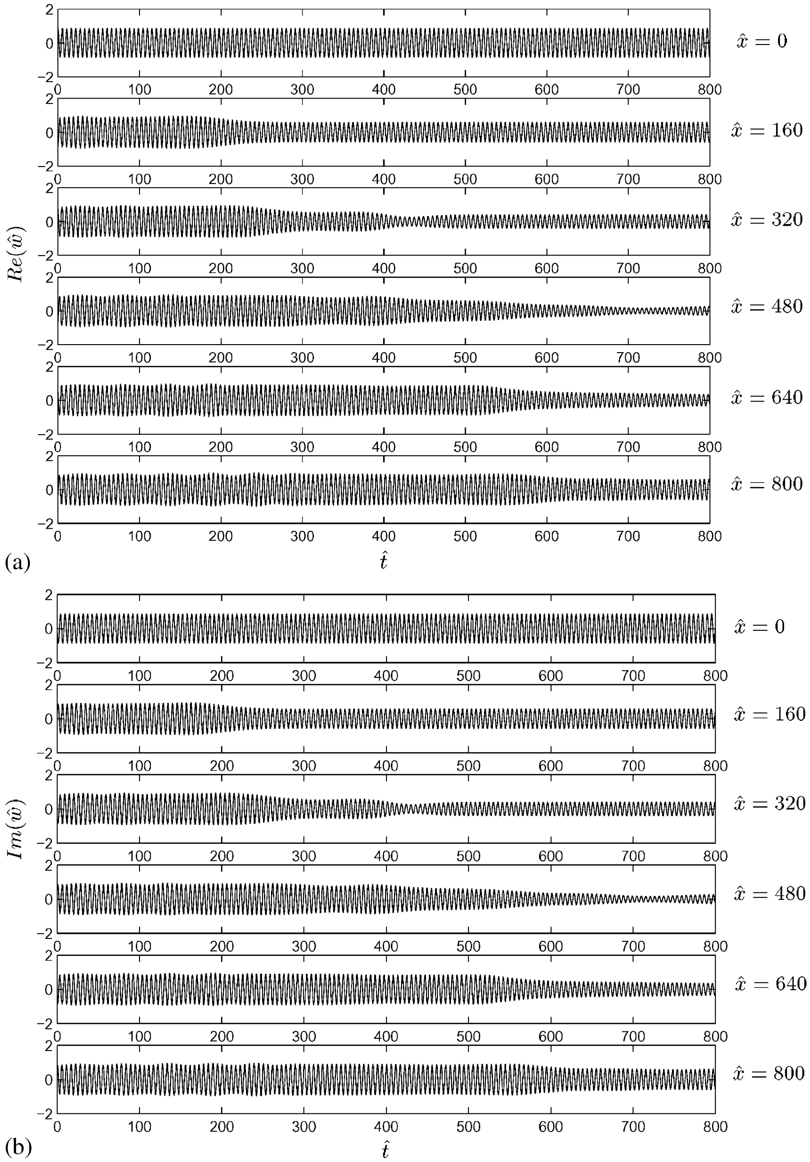

Fig. 3. Real and imaginary parts of $\hat{w}$ for non-dimensional wavenumber $K_0 = 1.0$ and $\varepsilon = 0.05$ at different observation points.

non-linearity ($\varepsilon = 0.01$), one can neglect the non-linear effects on the wave motion. On the other hand, for higher non-linearity ($\varepsilon = 0.05, 0.07$), it is seen that $\hat{w}$ decreases in amplitude as $\hat{t}$ increases. It decreases more as ε increases. Increasing ε has much more dramatic effect on the rotation $\hat{\psi}$. This is consistent with the behavior shown in Fig. 4. Initially, there is the linear dispersive behavior (beating), which is followed by a slowly varying time dependence. The amplitude reaches a higher steady value at long time.

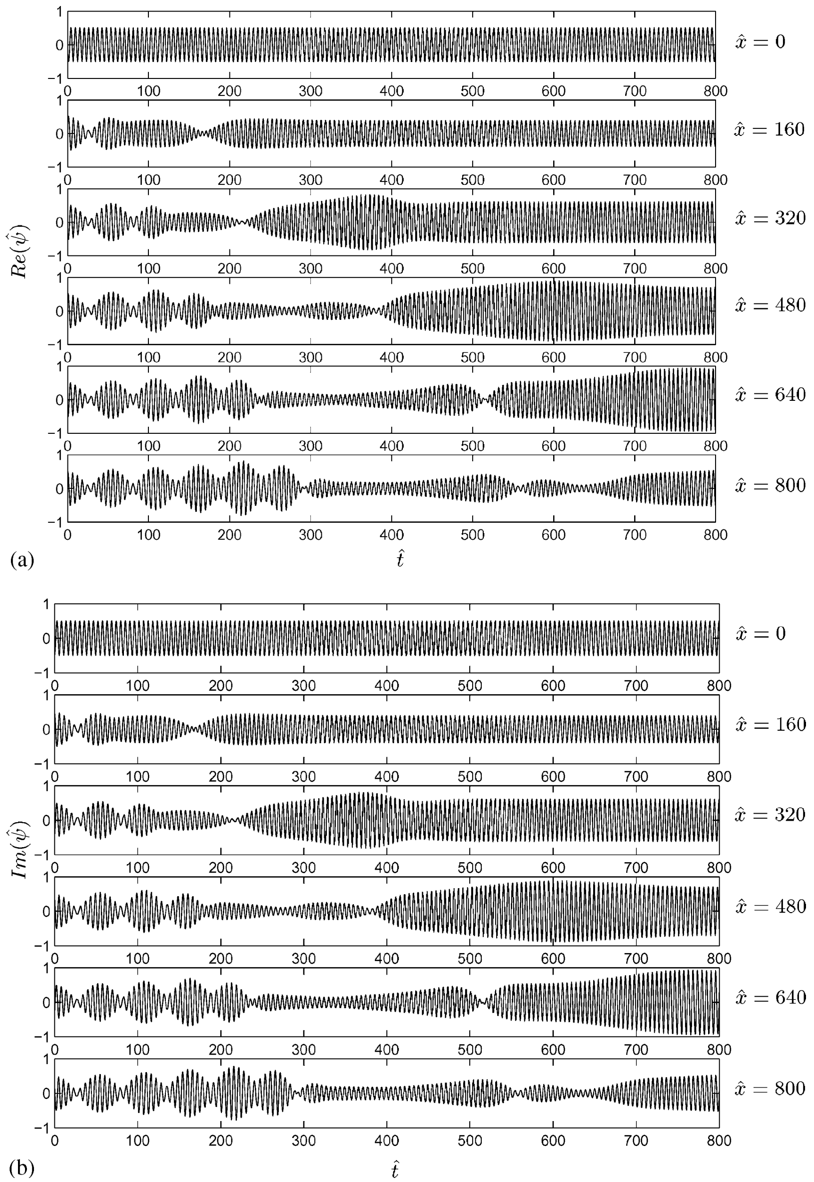

Fig. 4. Real and imaginary parts of $\hat{\psi}$ for non-dimensional wavenumber $K_0 = 1.0$ and $\varepsilon = 0.05$ at different observation points.

In Fig. 6, plots of the real parts of $\hat{w}$ and $\hat{\psi}$ are shown for different wavenumber $K_0 = 0.5, 1.0, 1.5$. Traveling wave profiles are calculated at a distance of $\hat{x} = 200$, when the non-linear parameter $\varepsilon = 0.05$. In this figure, the attenuation in $\hat{w}$ is clearly seen and non-linearity has more influence on the wave motion at high wavenumber and frequency. In Fig. 6(b), wave amplification is observed in $\hat{\psi}$ as the wavenumber increases. It is seen that the rotation is negligible at small

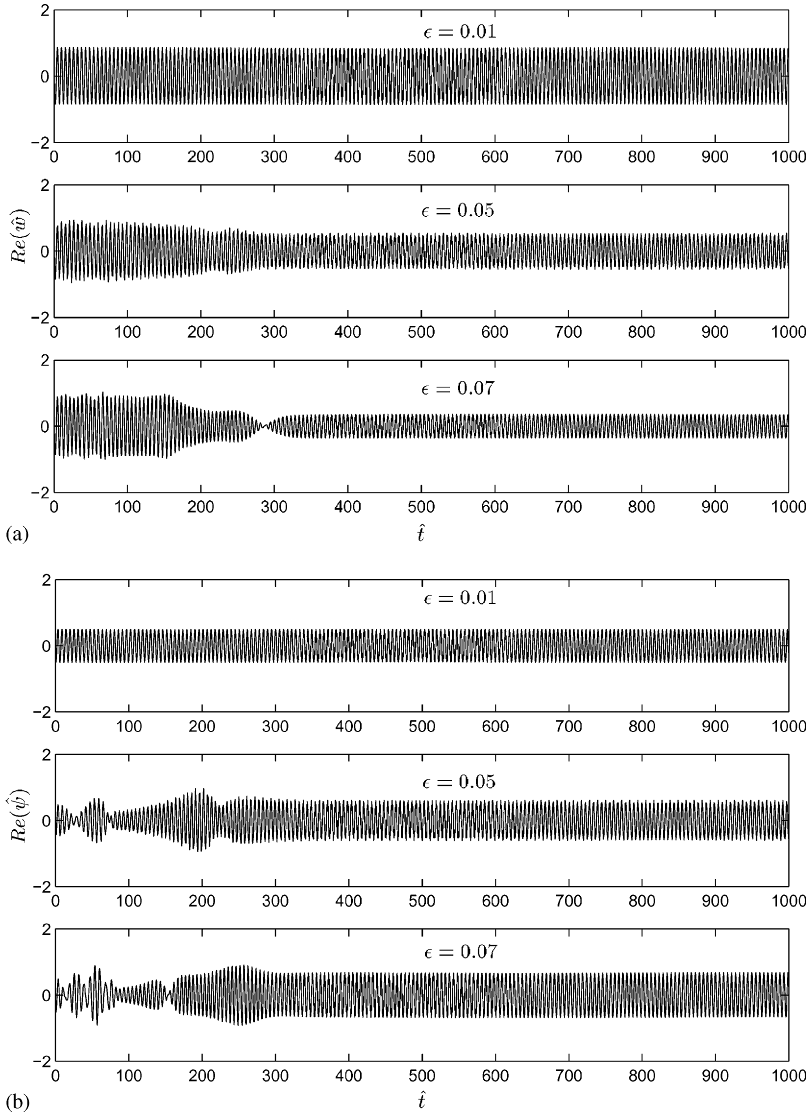

Fig. 5. Real parts of the amplitude components for different ε calculated at $\hat{x} = 200$ for non-dimensional wavenumber $K_0 = 1.0$.

wavenumbers. But as the wavenumber increases, the rotation is amplified and exhibits distinctly different behavior in three time regions: short, intermediate, and long.

Fig. 7 shows the effect of geometric and material non-linearities separately on the wave amplitude calculated at $\hat{x} = 200$ for $\varepsilon = 0.05$. Geometric non-linearity shows large attenuation in $\hat{\psi}$ component, but negligible effect on the $\hat{w}$ component. However, the combined effect of both has significant influence on $\hat{\psi}$. There is an increase in amplitude at long time and it has different characteristics at short, intermediate, and long times. In contrast, material non-linearity is seen to attenuate the transverse displacement amplitude.

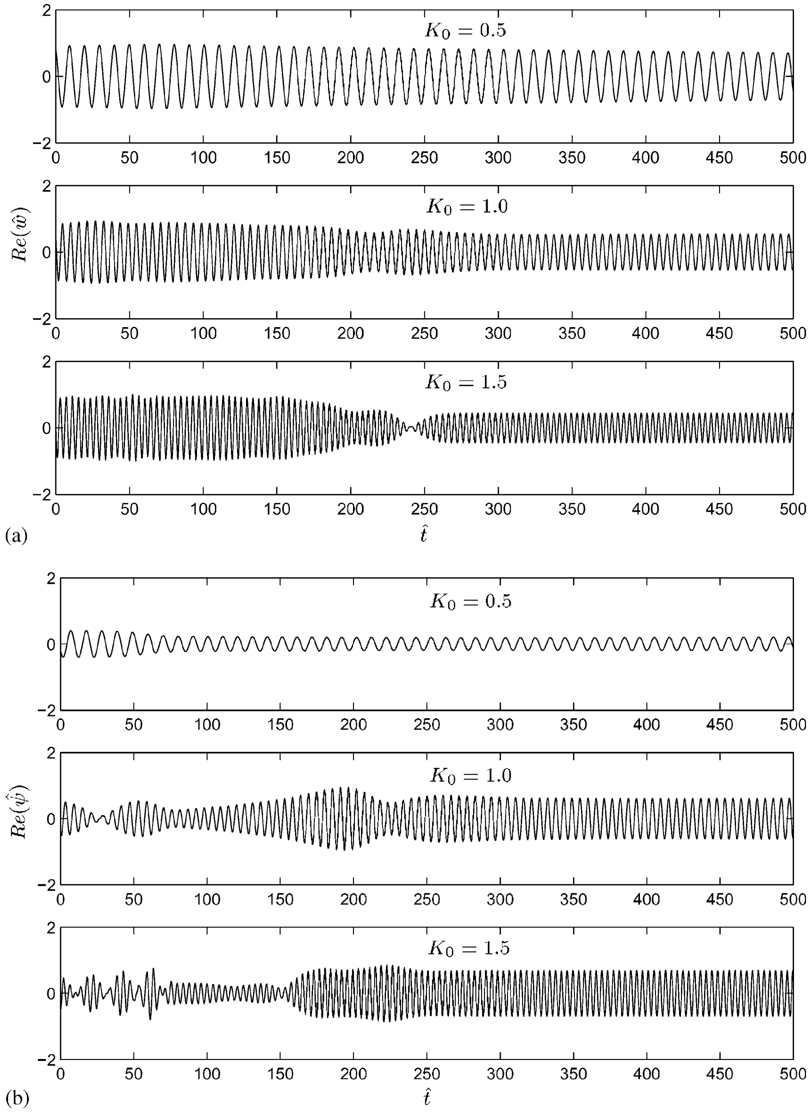

Fig. 6. Real parts of $\hat{w}$ and $\hat{\psi}$ at different wavenumbers K_0 calculated at $\hat{x} = 200$ for $\varepsilon = 0.05$.

4. Conclusions

Non-linear flexural waves in an elastic thin plate are modelled using Mindlin plate theory. The non-linearity is modelled by considering non-linear Von Karman strains and by including quadratic strains in the constitutive equation for the elastic material. An asymptotic solution is obtained using the method of multiple scales, and an evolution equation, describing the amplitude dependence on the slow time and space co-ordinates, is also derived and solved numerically. Combined effects of geometric and material non-linearities have been studied. Results show that the transverse displacement amplitude decreases with time, whereas the rotational component

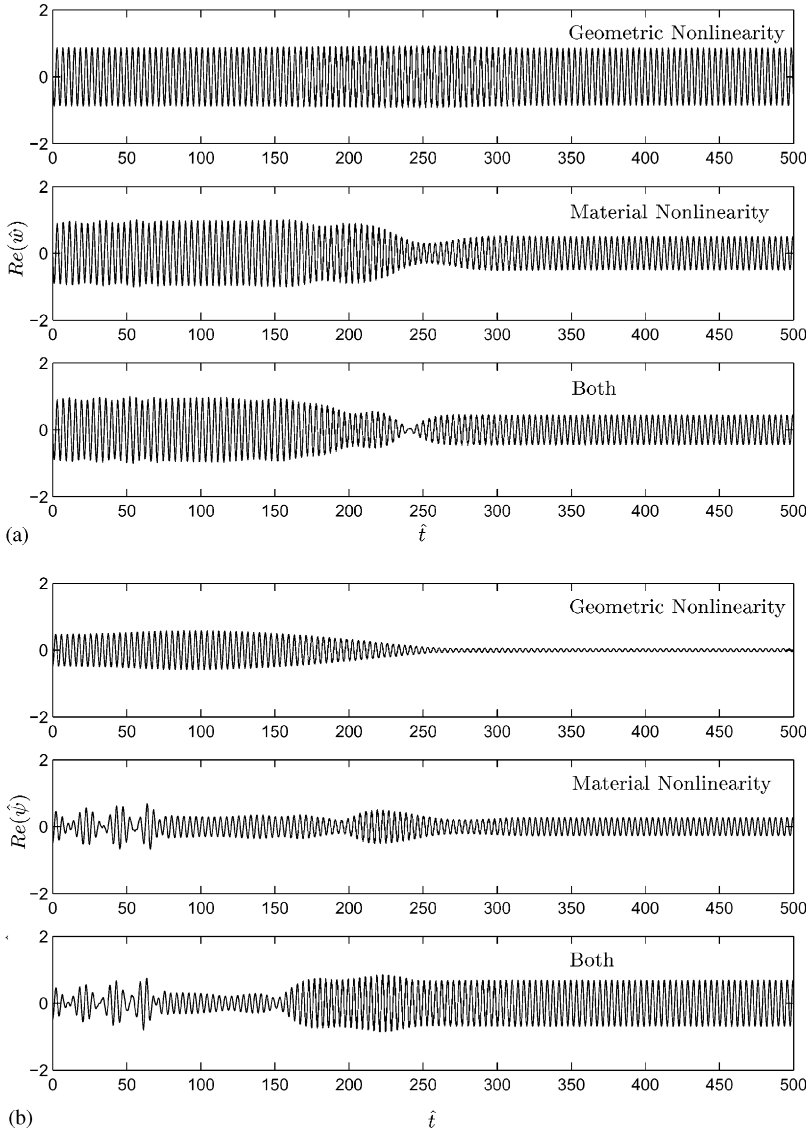

Fig. 7. Non-linearity effects on the real parts of $\hat{w}$ and $\hat{\psi}$ at $K_0 = 1.5$ calculated at $\hat{x} = 200$ for $\varepsilon = 0.05$.

shows an increase in amplitude. The latter also has distinctly different behaviors at short, intermediate and long times.

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